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The Geometry of Differential Elements of the Second Order with Respect to the Group of all Point Transformations.

By Edward Kasner.

A simple and familiar example of an expression which is unaltered by arbitrary point transformation of the plane is given by the anharmonic ratio of the tangents to four curves passing through a common point. It is understood that the functions defining the transformation

$$X = \phi(x, y), \quad Y = \psi(x, y) \tag{1}$$

possess first derivatives and that the Jacobian

$$J \equiv \phi_x \psi_y - \phi_y \psi_x \tag{2}$$

does not vanish at the point in question. The invariant property here results from the fact that, whatever the form of ϕ and ψ, the lineal elements at a point undergo linear transformation, namely

$$Y' = \frac{\psi_x + y' \psi_y}{\phi_x + y' \phi_y}.$$

In this paper we obtain absolute invariants involving the curvatures as well as the directions of curves passing through a point, *i. e.*, invariants of differential elements of the second order. The transformations of y', y'' are no longer linear, but cubic, and give rise to the eight-parameter group discussed in §1. The invariants are obtained from the infinitesimal generators of this group in §5. They may also be obtained synthetically by means of a fundamental relation, established in §2, between the geometry of differential elements considered with respect to arbitrary point transformations and projective geometry in space of four dimensions. This relation suggests a rational classification of differential equations of the second order (§3). It also furnishes a geometric interpretation for the new invariants obtained (§6). In the last section (§7) the invariants for conformal transformation, which are not all expressible as angles, are obtained by a very simple relation to similitude transformations.

§1.—*The Group* G_8.

Consider a bundle of elements, *i. e.*, the ∞^2 differential elements of the second order at a given point. An element is determined by the values of y', y''. By a point transformation the bundle is turned into a bundle and the coordinates of the new elements are

$$Y' = \frac{\psi_x + y'\psi_y}{\phi_x + y'\phi_y}, \quad Y'' = \frac{\alpha + \beta y' + \gamma y'^2 + \delta y'^3 + Jy''}{(\phi_x + y'\phi_y)^3}, \tag{3}$$

where $\alpha, \beta, \gamma, \delta$ involve the second derivatives of ϕ and ψ.* The coefficients are numbers since the point x, y is fixed and only y', y'' vary. It will be convenient to use the simplified notation

$$u = y',\ v = y'';\ U = Y',\ V = Y''.$$

Our transformation is then of the form

$$U = \frac{au + b}{cu + d}, \quad V = \frac{\alpha + \beta u + \gamma u^2 + \delta u^3 + (ad - bc)v}{(cu + d)^3}, \tag{4}$$

This cannot degenerate since it is assumed that

$$J = ad - bc \neq 0.$$

The constants a, b, c, d, α, β, γ, δ are arbitrary and essential.

The infinite group of arbitrary point transformations thus induces on a bundle of elements the eight-parameter continuous group (4), *which we shall refer to as* G_8.

The eight infinitesimal generators of the group are found to be

$$\frac{\partial}{\partial u},\ \frac{\partial}{\partial v},\ u\frac{\partial}{\partial u},\ v\frac{\partial}{\partial v},\ u^2\frac{\partial}{\partial u} + 3uv\frac{\partial}{\partial v},\ u\frac{\partial}{\partial v},\ u^2\frac{\partial}{\partial v},\ u^3\frac{\partial}{\partial v}. \tag{5}$$

§2.—*Representation in Four-dimensional Space.*

It will be noted that the numerator and denominator of V in (4) are linear with respect to u, u^2, u^3, v. This suggests the representation of each element

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* It is assumed that these second derivatives exist and that J does not vanish at the point considered.

(u, v) of our bundle by means of a point in four space R_4 with the homogeneous coordinates*

$$z_1 : z_2 : z_3 : z_4 : z_5 = 1 : u : u^2 : u^3 : v \tag{6}$$

The ∞^2 points in R_4 representing the ∞^2 elements then constitute the manifold defined by

$$z_1 z_3 - z_2^2 = 0, \quad z_2 z_3 - z_1 z_4 = 0, \quad z_3^2 - z_2 z_4 = 0. \tag{7}$$

This is a cubic cone S_3 with its vertex at the point $(0, 0, 0, 0, 1)$; its section in the space $z_5 = 0$ is a twisted cubic.

The transformation (4) now may be written in the following linear form:

$$\left.\begin{aligned}
Z_1 &= d^3 z_1 + 3\,cd^2 z_2 + 3c^2 d z_3 + c^3 z_4 \\
Z_2 &= b^2 d z_1 + d(ad + 2bc) z_2 + c(bc + 2ad) z_3 + ac^2 z_4 \\
Z_3 &= bd^2 z_1 + b(bc + 2ad) z_2 + a(ad + 2bc) z_3 + a^2 c z_4 \\
Z_4 &= b^3 z_1 + 3ab^2 z_2 + 3a^2 b z_3 + a^3 z_4 \\
Z_5 &= \alpha z_1 + \beta z_2 + \gamma z_3 + \delta z_4 + (ad - bc) z_5
\end{aligned}\right\} \tag{8}$$

This leaves the relations (4) invariant.

Thus to each transformation (4) *of the bundle of elements corresponds in R_4 a collineation for which the cubic surface S_3 is invariant.*

It is easy to show in fact that a cubic cone in R_4 admits no larger continuous group of collineation. The general collineation of R_4 involves 24 constants. Since the vertex of the cone must be a fixed point, these are reduced to 20. Furthermore, the twisted cubic in the plane $z_5 = 0$ must be converted into one of the ∞^4 twisted cubics on the cone. Since the total number of twisted cubics in R_4 is ∞^{16}, this imposes $16 - 4 = 12$ restrictions. Hence the automorphic group in question contains precisely $20 - 12 = 8$ parameters.

We have thus obtained the following principle:

The geometry of a bundle of differential elements of the second order in a plane, with respect to arbitrary point transformation, is equivalent to projective geometry on a cubic cone in a space of four dimensions.

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*The z's may of course be considered directly as a system of redundant coordinates for defining an element of a bundle. This is the natural system for the group of all point transformations. For the projective group of the plane, an appropriate system has been introduced by E. Study (Leipziger Berichte, 1901).

§3.—*Classification of Differential Equations of the Second Order.*

This principle suggests the classification of differential equations of the second order which are algebraic in y', y'', that is in u, v, according to the degree of the corresponding equation expressed in the z coordinates. To an equation of first degree

$$A_1 z_1 + A_2 z_2 + A_3 z_3 + A_4 z_4 + A_5 z_5 = 0,$$

corresponds in terms of u, v an equation

$$A_1 + A_2 u + A_3 u^2 + A_4 u^3 + A_5 v = 0.$$

If now the coefficients are taken as arbitrary functions of x, y, we have the differential equations of the form

$$Ey'' = Ay'^3 + By'^2 + Cy' + D. \tag{9}$$

This is the type which has been extensively studied by Lie, R. Liouville, and Tresse. It includes the differential equation of the geodesics on any surface.

The next case, corresponding to an equation of second degree in the z's, leads to the equations

$$\begin{aligned} &Ay''^2 + (B_1 + B_1 y' + B_2 y'^2 + B_3 y'^3)y'' \\ &\qquad + (C + C_1 y' + C_2 y'^2 + C_3 y'^3 + C_4 y'^4 + C_5 y'^5 + C_6 y'^6) = 0 \end{aligned} \tag{10}$$

where the coefficients are arbitrary functions of x, y.

In general we define the *rank* of a differential equation which is algebraic with respect to y' and y'' as the degree of the corresponding z equation.

The rank of a differential equation is invariant under arbitrary point transformation.

The equations of second rank, given by (10), involve 11 arbitrary coefficients instead of the 14 contained in a general quadric in R_4. This is of course due to the three quadratic identities (7). The most general equation of rank r involves

$$\frac{(r+1)(r+2)(r+3)(r+4)}{1\,.\,2\,.\,3\,.\,4} - 3\,\frac{(r-1)\,r\,(r+1)(r+2)}{1\,.\,2\,.\,3\,.\,4} - 1$$

arbitrary functions of x, y.

Equality of rank is necessary, but not sufficient for the equivalence of two differential equations under point transformation. Thus, among the equations of second rank there is an invariant species defined by $A = 0$; this arises when the quadric manifold in R_4 passes through the vertex of the cone. Subspecies may then be distinguished according to the number of generators of the cone which lie in the quadric. Thus for two generators we find the type

$$(B+B_1y')\,y''+(C+C_1y'+C_2y'^2+C_3y'^3+C_4y'^4)=0, \tag{11}$$

which is thus invariant under point transformation.

In order that two differential equations of the second order shall be equivalent under point transformation, it is NECESSARY *that the corresponding curves on the fundamental cone in R_4 shall be projectively equivalent.**

§4.—*The Transformation of Centres.*

To interpret the equations of various ranks geometrically, without leaving the plane, consider for each element (u, v) the corresponding centre of curvature

$$x_0=-\frac{u(1+u^2)}{v},\quad y_0=\frac{1+u^2}{v}. \tag{12}$$

A point transformation, operating on the elements according to (4), induces the following transformation of the centres:

$$\left.\begin{aligned} X_0&=\frac{(ax_0-by_0)\,\{(ax_0-by_0)^2+(cx_0-dy_0)^2\}}{\alpha y_0^3-\beta x_0y_0^2+\gamma x_0^2y_0-\delta x_0^3+(ac-bd)\,(x_0^2+y_0^2)}\\ Y_0&=\frac{-(cx_0-dy_0)\,\{(ax_0-by_0)^2+(cx_0-dy_0)^2\}}{\alpha y_0^3-\beta x_0y_0^2+\gamma x_0^2y_0-\delta x_0^3+(ac-bd)\,(x_0^2+y_0^2)}\end{aligned}\right\} \tag{13}$$

The centres thus undergo a group of cubic Cremona transformations.

An equation of first rank (9) establishes a relation between x_0 and y_0 of the form

$$E(x_0^2+y_0^2)=-Ax_0^3+Bx_0^2y_0-Cx_0y_0^2+Dy_0^3. \tag{14}$$

In case of equations of first rank, the locus of the centres of curvature of the ∞^1 *integral curves passing through a fixed point is a cubic curve of type* (14).†

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*The invariants of differential equations of the second order have been determined by Tresse in the Preischriften Jablon. Gesell. zu Leipzig, vol. 32, 1896. Even the equations of first rank have invariants, and are therefore not all equivalent.

†Such a cubic curve is characterized by the fact that the fixed point (the origin) is a double point and the two tangents are the minimal lines.

For the general equation of rank r, the central locus is a special curve of order $3r$. If the central locus in the general curve of order n, the corresponding differential equation is a special equation of rank n.

As remarked above, the differential equation of the geodesics on any surface expressed in arbitrary Gaussian coordinates is of first rank. Therefore

If a surface is represented point by point on a plane, the geodesics through a point are pictured by curves whose centres of curvature at the common point lie on a cubic curve of type (14).*

In particular, a pencil of straight lines is converted by a point transformation into a system of curves for which the central locus is one of cubics considered.

§5.—*Absolute Invariants.*

Consider any number of elements

$$(u_1, v_1), \quad (u_2, v_2), \quad \ldots \quad , \quad (u_n, v_n).$$

A function of these $2n$ arbitrary quantities is an absolute invariant for all point transformations provided it admits the group G_8 of §1, or, what is the equivalent, the eight infinitesimal transformations (5). We thus have the following system of linear partial differential equations:

$$\left.\begin{array}{c} \sum \frac{\partial f}{\partial u_k} = 0, \quad \sum \frac{\partial f}{\partial v_k} = 0, \quad \sum u_k \frac{\partial f}{\partial u_k} = 0, \quad \sum v_k \frac{\partial f}{\partial v_k} = 0, \\ \sum \left(u_k^2 \frac{\partial f}{\partial u_k} + 3 u_k v_k \frac{\partial f}{\partial v_k} \right) = 0, \\ \sum u_k \frac{\partial f}{\partial v_k} = 0, \quad \sum u_k^2 \frac{\partial f}{\partial v_k} = 0, \quad \sum u_k^3 \frac{\partial f}{\partial v_k} = 0, \end{array} \right\} \tag{15}$$

where the summations are extended from $k = 1$ to $k = n$.

The first two of the equations (15) show that f involves only the differences of the u's and the differences of the v's; while the next two show that f is homogeneous in these differences. Hence f is expressible in terms of the $2n - 4$ quantities

$$\xi_i = \frac{u_i - u_1}{u_2 - u_1}, \quad \eta_i = \frac{v_i - v_1}{v_2 - v_1} \qquad (i = 3, 4, \ldots, n). \tag{16'}$$

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* If the representation is conformal, the central locus is a straight line. This holds indeed for a more extensive class of representations. See Bull. Amer. Math. Soc., vol. 12, 1906, p. 430.

If we introduce these as the independent variables, the last four equations of the system (15) become

$$\left.\begin{array}{l} \Sigma\xi_i(\xi_i - 1) f_{\xi_i} + 3\Sigma\eta_i(\xi_i - 1) f_{\eta_i} = 0, \\ \Sigma(\eta_i - \xi_i) f_{\eta_i} - 0, \quad \Sigma\xi_i(\xi_i - 1) f_{\eta_i} = 0, \quad \Sigma\xi_i^2(\xi_i - 1) f_{\eta_i} = 0, \\ \hfill (i = 3, \ldots\ldots, n) \end{array}\right\} \qquad (16)$$

Integrating the first of these equations, we find that f is a function of the $2n - 5$ quantities

$$\left.\begin{array}{ll} r_i = \dfrac{\xi_3(\xi_i - 1)}{\xi_i(\xi_3 - 1)} & (i = 4, \ldots\ldots, n) \\ s_i = \dfrac{\eta_i}{\xi_i^3} & (i = 3, \ldots\ldots, n) \end{array}\right\} \qquad (17')$$

The remaining equations (16) then become

$$\left.\begin{array}{l} \sum_4^n (s_i - 1) f_{s_i} = 0, \\ f_{s_3} + \sum_4^n r_i f_{s_i} = 0, \\ f_{s_3} + \sum_4^n r_i^2 f_{s_i} = 0. \end{array}\right\} \qquad (17)$$

The first of these equations shows that we may take as the independent variables the $2n - 6$ quantities

$$\left.\begin{array}{ll} \sigma_i = \dfrac{s_i - 1}{s_3 - 1} & (i = 4, \ldots\ldots, n) \\ r_i & \end{array}\right\} \qquad (18')$$

The other equations then reduce to

$$\left.\begin{array}{l} \sum_4^n (\sigma_i - r_i) f_{s_i} = 0, \\ \sum_4^n (\sigma_i - r_i^2) f_{s_i} = 0. \end{array}\right\} \qquad (18)$$

The next step in the solution brings in the independent variables

$$\left.\begin{array}{ll} \tau_i = \dfrac{\sigma_i - r_i}{\sigma_4 - r_4} & (i = 4, \ldots\ldots, n), \\ r_i & \end{array}\right\} \qquad (19')$$

and reduces the system to the single equation

$$\sum_{5}^{n} \{ r_i - r_i^2 + \tau_i (r_4^2 - r_4) \} f_{\tau_i} = 0. \tag{19}$$

Integrating this, we find that f is a function of

$$\left.\begin{array}{ll} \Omega_i = \dfrac{(r_4^2 - r_4)\,\tau_i - (r_i^2 - r_i)}{(r_4^2 - r_4)\,\tau_5 - (r_5^2 - r_5)} & (i = 6, \ldots, n) \\ r_i & (i = 4, \ldots, n) \end{array}\right\} \tag{20}$$

Our result then is that

All (absolute) invariants, with respect to the infinite group of point transformations, of n arbitrary differential elements of second order $(u_1, v_1), \ldots, (u_n, v_n)$ *may be expressed as functions of* $2n - 8$ *invariants, namely:* $n - 3$ *anharmonic ratios*

$$r_4, r_5, \ldots, r_n,$$

depending upon only the directions of the elements, and $n - 5$ *of new type*

$$\Omega_6, \Omega_7, \ldots, \Omega_n,$$

which involve the curvatures.

The new type presents itself then only when there are six (or more) elements.*

The system of invariants given is complete not merely in the general functional sense, but also in a special algebraic sense. A detailed discussion, by means of a canonical representation, yields in fact the conclusion:

All RATIONAL *invariants, i. e., all invariants which involve* $u_1, v_1, \ldots, u_n, v_n$ *rationally, are expressible as* RATIONAL *functions of the fundamental invariants of types* r *and* Ω.

§6.—*Geometric Interpretation.*

In order to obtain a geometric interpretation for our absolute invariants, we make use of the corresponding projective geometry in R_4

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* In his "Théorie des Invariants Universelles" (Jour. de l'École Polytechnique, ser. 2, vol. 4, 1898), P. Rabut attempts to determine all the invariants (relative as well as absolute) of differential elements in general. His method, however, seems to be special, as he does not obtain the above type Ω. In fact, one of his conclusions is that elements whose tangents are all distinct have no absolute invariants beyond those of the familiar type r.

If the group considered is the general projective group of R_4, and points are taken in arbitrary position, then no absolute invariant occurs for fewer than 7 points. In the case of 7 points, we may pass 3-flats through each of the quadruples 1234, 1235, 1236, 1237; these are members of a pencil and thus have an anharmonic ratio which is, of course, an absolute invariant. Our problem, however, is to find the invariants of points situated on the cone S_3 with respect to the special collineation group (8). Here invariants arise for four points. For these determine four generators of the cone, which have an anharmonic ratio, namely, the anharmonic ratio of the four points in which these generators intersect any one of the twisted cubics on S_3. This invariant is of course equivalent to r_4.

Consider now the case of six points on the surface. If we adjoin to these the vertex, (0, 0, 0, 0, 1), of the cone, which is of course invariant for G_8, we have seven points, and so the general projective geometry applies. If the vertex is one of the three points through which 3-flats are passed, we obtain simply one of the r invariants. But if this is not the case, the invariant is of new type.

We can now transfer these results to the plane. Any four elements of a bundle determine an equation of first rank; hence the four centres of curvature determine a cubic curve of type (14).* If now we have six elements at the common point C_0 with centres denoted by $C_1, C_2, C_3, C_4, C_5, C_6$, we may construct the proper cubics determined by the quadruples

$$C_1\,C_2\,C_3\,C_4, \quad C_1\,C_2\,C_3\,C_5, \quad C_1\,C_2\,C_3\,C_6$$

and the degenerate cubic determined by

$$C_1\,C_2\,C_3\,O,$$

the last consisting of the straight lines connecting C_1, C_2, C_3 with the fixed point O. These cubics are members of a pencil, and hence have an anharmonic ratio I_6, which may be represented concretely by the anharmonic ratio of the tangents at say C_1 to the proper cubics and the line $C_1\,O$.

This absolute invariant I_6 is equivalent to Ω_6; either one may be expressed as a rational function of the other and of invariants of type r. The explicit expressions may be obtained by means of a canonical form, but are not of any particular interest. The complete system obtained in §5 may be replaced by that composed of the types r and I.

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*The cubic degenerates only when two of the elements have a common tangent, or when one of the elements is cuspidal. These cases are explicitly excluded.

§7.—*Conformal Invariants.*

If the point transformation (1) is conformal, the coefficients appearing in the element transformation (4) are connected by the relations

$$a-d=0,\quad b+c=0,$$
$$\alpha-\gamma=0,\quad \beta-\delta=0.$$

Hence the conformal transformations give rise to a four-parameter sub-group G_4:

$$U=\frac{au+b}{a-bu},\quad V=\frac{(\alpha+\beta u)(1+u^2)+(a^2+b^2)v}{(a-bu)^3}. \tag{21}$$

The infinitesimal generators are

$$v\frac{\partial}{\partial v},\quad (1+u^2)\frac{\partial}{\partial v},\quad u(1+u^2)\frac{\partial}{\partial v},\quad (1+u^2)\frac{\partial}{\partial u}+3uv\frac{\partial}{\partial v}. \tag{22}$$

The invariants of any number of elements with respect to conformal transformations might be calculated directly from these infinitesimal transformations; but a simple principle, now to be established, will yield them synthetically.

We observe first that if (21) is expressed in terms of the corresponding centres of curvature, we obtain instead of the cubic transformations (13), the linear transformations

$$X_0=\frac{(a^2+b^2)(ax_0-by_0)}{-\beta x_0+\alpha y_0+a^2+b^2},\quad Y_0=\frac{(a^2+b^2)(bx_0+ay_0)}{-\beta x_0+\alpha y_0+a^2+b^2}. \tag{23}$$

This projective group is composed of those collineations of the plane which leave invariant the origin and the minimal lines passing through it.

If we take the dual of these transformations with respect to the unit circle about the origin, the new transformations leave invariant the line at infinity and the two circular points upon it, and hence are similitude transformations. Hence the conformal group (21) may be related to the similitude group.

We thus obtain a correspondence between the ∞^2 *elements of our bundle and the* ∞^2 *straight lines of the plane. The line corresponding to any given element is the polar of its centre of curvature with respect to the unit circle.* If the differential coordinates of the element are (u, v), the corresponding straight line is

$$u(1+u^2)\,x-(1+u^2)y+v=0.$$

When the elements are operated on by any conformal transformation, the induced transformation of the straight lines is simply a similitude transformation.

Hence the conformal invariants of any number of elements may be obtained from the similitude invariants of the same number of straight lines.

For two straight lines, the (absolute) invariant is their angle. This of course is also the angle between the elements. For three lines no new type occurs. For four lines there are only three independent angles, while the number of independent invariants is four; the new type is say the ratio of the segments formed by the intersections of three of the lines on the fourth line. For n lines we have $n-1$ angles and $n-3$ independent ratios.*

With respect to the infinite group† *of conformal transformations of the plane, n arbitrary curvature elements of a bundle have $2n-4$ independent absolute invariants; of these $n-1$ are the angles between successive elements and $n-3$ are of new type.*

The latter may be expressed either as ratios, by means of the corresponding straight lines, or as cross-ratios by means of the centres of curvature. If four elements at the common point O have the centres C_1, C_2, C_3, C_4, the new invariant is equivalent to the anharmonic ratio of the lines connecting say C_1 with C_2, C_3, C_4, and O.

The classification of differential equations given in §3 of course remains valid for the conformal sub-group. Equations of the same rank may furthermore be divided into species according to the order of the central loci. When the loci are straight lines, the equations are of the type

$$y'' = (My' + N)(1 + y'^2), \tag{24}$$

which is conformally distinct from the other equations of first rank, though not distinct in the general point group. It may be shown, indeed, that the conformal transformations are the only ones leaving the type (24) invariant.

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* It has been assumed that the conformal equations are of the direct type (not reversing angles) so that the groups considered are continuous. In the reverse type we obtain, instead of (23), a projective transformation which interchanges the minimal lines at 0; the corresponding similitude is then also of the reverse type.

† An examination of the six-parameter group of circular transformations yields the same results. Cf. the author's paper "A relation between the circular and projective transformations of the plane" (Annals of Mathematics, Vol. 5, 1904, pp. 99–104). The invariant described in the closing sentence on page 104 is independent of the angles between the circles.